Name:____________________________

MA 8643 Section 01 *Practice Midterm Exam* *November 19, 2019*

Follow the instructions for each question and show enough of your work so that I can follow your thought process. If I can't read your work, answer or there is no justification to a solution you will receive little or no credit!

1. Let $X = \{a, b, c\}$. On the collection of subsets $\mathcal{M} = \{\emptyset, \{a\}, \{a, b\}\}$, define the set function $n : \mathcal{M} \to [0, \infty]$ by $n(\emptyset) = 0$, $n(\{a\}) = 2$ and $n(\{a, b\}) = 1$. Find an outer measure μ^* on X such that $\mu^*|_{\mathcal{M}} = n$ and find all of the μ^*-measurable sets of X. Be sure to justify.

2. Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall the symmetric difference of two subsets of X is

$$E_1 \Delta E_2 = (E_1 \setminus E_2) \cup (E_2 \setminus E_1) .$$

Show that if E_1 and E_2 are measurable and $\mu(E_1 \Delta E_2) = 0$, then $\mu(E_1) = \mu(E_2)$. Moreover show that if $(X, \mathcal{M}, \mu)$ is complete and $E_1 \in \mathcal{M}$, then $E_2 \in \mathcal{M}$ provided $\mu(E_1 \Delta E_2) = 0$.

3. Let $(X, \mathcal{M}, \mu)$ be a measure space. Show that an extended real-valued function, f, on X is measurable if and only if $f^{-1}\{\infty\}$ and $f^{-1}\{-\infty\}$ are measurable and so is $f^{-1}(E)$ for every Borel set E of real numbers.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space. Consider two extended real-valued measurable functions f and g on X that finite a.e. on X. Define the following set:

$$X_0 = \{x \in X : f, g < \infty\} .$$

Show that X_0 is measurable and $\mu(X \setminus X_0) = 0$.

5. Let $(X, \mathcal{M}, \mu)$ be a measure space, not necessarily σ-finite. Prove that for any measurable $f : X \to [0, \infty]$, not necessarily integrable,

$$\int_X f \, d\mu = \int_0^\infty \mu\{x \in X : f(x) > t\} \, dt .$$

6. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}$ be a sequence of integrable functions on X that is uniformly integrable. Suppose that $f_n \to f$ pointwise a.e. on X and f is measurable. Assume the measure space has the property that for each $\varepsilon > 0$, X is the union of a finite collection of measurable sets, each of measure at most ε. Prove that f is integrable over X.

7. Let $(X, \mathcal{M}, \mu)$ be a finite measure space, $\{E_k\}_{k=1}^n$ a collection of measurable sets, and $\{c_k\}_{k=1}^n$ a collection of real numbers. For $E \in \mathcal{M}$, define

$$\nu(E) = \sum_{k=1}^{n} c_k \cdot \mu(E \cap E_k)\,.$$

Prove that ν is absolutely continuous with respect to μ and compute the Radon-Nikodym derivative.

8. Let μ and ν be measures on the measurable space $(X, \mathcal{M})$ and define $\lambda = \mu + \nu$. Let the nonnegative function f on X be measurable with respect to $(X, \mathcal{M})$. Show that f is integrable over X with respect to λ if and only if it is integrable over X with respect to both μ and ν. Moreover prove that if f is integrable over X with respect to λ, then

$$\int_E f\, d\lambda = \int_E f\, d\mu + \int_E f\, d\nu$$

for all $E \in \mathcal{M}$.

9. Let $\mathcal{M}$ be the collection of Lebesgue measurable sets in $[0, 1]$. Let m and η be the Lebesgue and counting measures on $\mathcal{M}$ respectively. Prove that m is finite and $m << \eta$, but there is no such function f such that

$$m(E) = \int_E d\eta$$

for all $E \in \mathcal{M}$.